A.D.M College for Women (Autonomous)

Nagapattinam

STUDY MATERIAL

Differential Calculus and Trigonometry

I B. Sc Mathematics

Semester: I ***Code: BMA***

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SUCCESSIVE DEFFERENTIATION

The n^{th} derivative.

For certain functions a general expression involving n may found for the n^{th} derivative. The usual plan is to find a number of successive derivatives, as many as may be necessary to discover their law of formation and then by induction write down n^{th} derivative.

Examples: If $y = e^{ax}$;then $\frac{dy}{dx} = ae^{ax}; \frac{d^2y}{dx^2} = a^2e^{ax} \ldots \frac{d^ny}{dx^n} = a^ne^{ax}$.

Standard results:

(1) If $y = (ax+b)^m$, then $D^n(ax+b)^m = (-1)^n n!\, a^n (ax+b)^{-n-1}$.

(2) If $y = \log(ax+b)$, then $y_n = (-1)^{n-1}(n-1)!\, a^n(ax+b)^{-n}$.

Solved Problems:

(1) Find y_n where $y = \frac{3}{(x+1)(2x-1)}$.

Solution:

Resolving into partial fractions, we obtain $y = \frac{2}{2x-1} - \frac{1}{x+1}$.

Thus $y_n = (-1)^n n! \left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}$.

(2) Find y_n where $y = \frac{x^2}{(x-1)^2(x+2)}$.

Solution:

Let $\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$.

Then we can easily find that $A = \frac{5}{9}; B = \frac{1}{3}; C = \frac{4}{9}$.

Hence $y_n = \frac{5}{9}\frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{1}{3}\frac{(n+1)!(-1)^n}{(x-1)^{n+2}} + \frac{4}{9}\frac{C}{(x+2)^{n+1}}$.

(3) Find y_n where $y = \frac{1}{x^2+a^2}$.

Solution: Let $y = \frac{1}{x^2+a^2} = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$

Then $y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$.

Trignometrical Transformation.

Solved Problems:

(1) Find the n^{th} differential coefficient of $\cos x\,.\cos 2x\,.\cos 3x$.

Solution:

$\text{Cos}\, x\,.\cos 2x\,.\cos 3x = \frac{1}{2}\cos 2x\,[\cos 4x + \cos 2x]$

$$= \frac{1}{4} + \frac{1}{4}[\cos 2x + \cos 4x + \cos 6x]$$

Hence $D^n(\cos x\,.\cos 2x\,.\cos 3x) =$

$$\frac{1}{4}\left\{2^n \cos\left(\frac{n\pi}{2} + 2x\right) + 4^n \cos\left(\frac{n\pi}{2} + 4x\right) + 6^n \cos\left(\frac{n\pi}{2} + 6x\right)\right\}.$$

(2) Find the n^{th} differential coefficient of $\cos^5\theta\,.\sin^7\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$. then $\frac{1}{x} = \cos\theta - i\sin\theta$.

Therefore $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$.

Hence by De Moivre's Theorem, we have,

$x^n = \cos n\theta + i\sin n\theta$ & $\frac{1}{x^n}$

$= \cos n\theta - i\sin n\theta$

so that $x^n + \frac{1}{x^n} = 2\cos n\theta$ and

$$x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

Thus $\cos^5\theta\,.\sin^7\theta = -2^{11}\cos^5\theta\,.\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$

$\mathrm{D^n}(\cos^5\theta\,.\sin^7\theta) = -\frac{1}{2^{11}}\left\{12^n \sin\left(\frac{n\pi}{2} + 12\theta\right) - 10^n.2\sin\left(\frac{n\pi}{2} + 10\theta\right) - 8^n.4\sin\left(\frac{n\pi}{2} + 8\theta\right) + 6^n.10\ \sin\left(\frac{n\pi}{2} + 6\theta\right) + 4^n\ 5\sin\left(\frac{n\pi}{2} + 4\theta\right) - 2^n.20\sin\left(\frac{n\pi}{2} + 2\theta\right)\right\}$

Leibnitz formula for the n^{th} derivative of a product:

Solved Problems:

(1) Find the n^{th} differential coefficients of $x^2 \log x$.

Solution:

Taking $v = x^2$ and $u = \log x$,

$$x^2 \log x = \frac{d^n}{dx^n}(\log x)x^2 + nc_1 \frac{d^{n-1}}{dx^{n-1}}(\log x)\frac{d}{dx}x^2 + nc_2 \frac{d^{n-2}}{dx^{n-2}}(\log x)\frac{d^2}{dx^2}x^2.$$

All the other terms will be zero and since the successive derivatives after the second derivative vanish.

$$\begin{aligned} D^n(x^2 \log x) &= \frac{(1)^{n-1}(n-1)!}{x^n}x^2 + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}2x + \frac{n(n-1)(-1)^{n-3}(n-3)!}{x^{n-2}} \\ &= \frac{(-1)^{n-2}(n-3)!}{x^{n-2}}2. \end{aligned}$$

(2) If $y = \sin(m \sin^{-1} x)$, prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y = 0$.

Solution:

Taking the n^{th} derivative of each term by Leibnitz theorem,

$$(1-x^2)y_{n+2} - nc_1 y_{n+1}(-2x) + nc_2 y_n(-2) = y_{n+1}x + nc_1 y_n - m^2 y_n.$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} + n(n-1)y_n = xy_{n+1} + ny_n - m^2 y_n.$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y = 0.$$

I. Curvature and Radius of Curvature:

(1) If a curve is defined by the parametric equation $x = f(\theta)$ & $y = \varphi(\theta)$, then the *curvature* is $\frac{1}{\rho} = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$.

(2) The cartesian formula for the *radius of curvature* is given by $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$.

(3) The radius of curvature when the curve is given in polar co-ordinates is given by $\rho = \frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}$.

Solved Problems:

(1) What is the radius of curvature for the curve $x^2 + y^2 = 2$ at the point $(1,1)$?

Solution:

Given: $x^2 + y^2 = 2$.

Differentiating the above equation, we get $4x^3 + 4y^3\frac{dy}{dx} = 0$.

Therefore $\frac{dy}{dx} = -\frac{x^3}{y^3}$.

Differentiating this once again, $\frac{d^2y}{dx^2} = \frac{3\left(x^3\frac{dy}{dx}-x^2y\right)}{y^4}$.

At the point $(1,1)$, $\frac{dy}{dx} = -1$ and $\frac{d^2y}{dx^2} = -6$.

Hence $\rho = \frac{\{1+(-1)^2\}^{3/2}}{-6} = -\frac{\sqrt{2}}{3}$.

(2) Show that the radius of curvature at any point of the catenary $y = \cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Solution:

Given $y = \cosh\frac{x}{c}$

Differentiating the above equation, we get $\frac{dy}{dx} = \sinh\frac{x}{c}$.

Differentiating this once again, we get $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}$.

$$\text{Hence } \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1+\left(\sinh\frac{x}{c}\right)^2\right\}^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{\left\{1 + \sinh^2 \frac{x}{c}\right\}^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{\cosh^3 \frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{y^2}{c}.$$

At any point (x, y),

$$\text{the normal} = y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} = y\cosh\frac{x}{c} = \frac{y^2}{c}.$$

Therefore, Radius of curvature=Length of the normal.

(3) Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a\cos\frac{\theta}{2}$.

Solution:

Given $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta) \;\&\; \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{d^2x}{d\theta^2} = -a\sin\theta \;\;\&\;\; \frac{d^2y}{d\theta^2} = a\cos\theta$$

Substituting in the formula (1), we get,

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

$$= \frac{a(1 + \cos\theta)a\cos\theta - \; a\sin\theta\,(-a\sin\theta)}{\{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}.$$

$$= \frac{a^2(1 + \cos\theta)}{a^3\{2(1 + \cos\theta)^2\}^{3/2}}$$

$$= \frac{2\cos^2 \theta/2}{a\left(4\cos^2 \theta/2\right)^{3/2}}$$

$$\Rightarrow \frac{1}{\rho} = \frac{1}{4a\cos\frac{\theta}{2}} \Rightarrow \rho = 4a\cos\frac{\theta}{2}.$$

(4) Find ρ at the point "t" of the curve $x = a(\cos t + t\sin t)$ and $y = a(\sin t + t\cos t)$.

Solution:

Given: $x = a(\cos t + t\sin t)$ and $y = a(\sin t + t\cos t)$.

$$\Rightarrow \frac{dx}{dt} = at\cos t \,\&\, \frac{dx}{dt} = at\sin t \Rightarrow$$

$$\frac{dy}{dx} = tan\ t \,\&\, \frac{d^2y}{dx^2} = \frac{1}{a\ \cos^3 t}.$$

Thus

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1 + tan^2\ t)}{\frac{1}{a\ \cos^3 t}} = at.$$

(5) Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$.

Solution:

Given $r = a(1 - \cos\theta)$.

Then $\frac{dr}{d\theta} = \text{a}\sin\theta \,\&\, \frac{d^2r}{d\theta^2} = a\cos\theta$.

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{8a^3 \sin^3 \theta/2}{6a^2 \sin^2 \theta/2}$$

$$= \frac{4}{3}a\sin\theta/2$$

$$= \frac{2}{3}\sqrt{2ar}.$$

I. Centre of Curvature and Evolute:

Let the centre of curvature of the curve $y = f(x)$ corresponding to the point $P(x, y)$ be X & Y. Then

$$X = x - \frac{y_1(1 + {y_1}^2)}{y_2} \quad \&$$

$$Y = y + \frac{1 + {y_1}^2}{y_2}.$$

The locus of the centre of curvature for a curve is called the evolute of the curve.

Solved Problems:

(1) Find the co-ordinates of the centre of curvature of the curve $xy = 2$ at the point $(2,1)$.

Solution:

Given: $xy = 2$

Now

$$\frac{dy}{dx} = -\frac{2}{x^2} \ \&$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$$\text{At } (2,1), \frac{dy}{dx} = -\frac{1}{2} \ \& \frac{d^2y}{dx^2} = \frac{1}{2}$$

$$\text{Thus } X = 2 + \frac{\left(1+\frac{1}{4}\right)\times\frac{1}{2}}{\frac{1}{2}} = 3\frac{1}{4} \ \&$$

$$Y = 1 + \frac{\left(1 + \frac{1}{4}\right)}{\frac{1}{2}} = 3\frac{1}{2}.$$

(2) Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$ is another cycloid.

Solution:

Given $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta) \ \& \ \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \cot\frac{\theta}{2} \quad \& \quad \frac{d^2y}{dx^2} = -\frac{1}{4a\sin^4 {\theta}/{2}}.$$

Thus

$$X = x + \frac{\left(1+\cos^2 {\theta}/{2}\right)\cot\frac{\theta}{2}}{\frac{1}{4a\sin^4 {\theta}/{2}}}$$

$$\Rightarrow X = a(\theta - \sin\theta).$$

$$\& \quad Y = y + \frac{1+\cot^2\frac{\theta}{2}}{-\frac{1}{4a\sin^4 {\theta}/{2}}}$$

$$\Rightarrow Y = a(1-\cos\theta).$$

II. Maxima and Minima:

Definition:

If a continuous function increases upto a certain and then decreases, that value is called a *maximum* value of the function. Similarly, If a continuous function decreases upto a certain and then increases, that value is called a *minimum* value of the function.

Solved Problems:

1) Find the maxima and minima of the function $2x^3 - 3x^2 - 36x + 10$.

Solution:

Let $f(x) = 2x^3 - 3x^2 - 36x + 10$

At the maximum or minimum, $f'(x) = 0$.

$$\Rightarrow f'(x) = 6(x-3)(x+2) = 0 \Rightarrow x = 3, -2$$

To distinguish between maximum or minimum, $f''(x) = 6(2x-1)$

When $x = 3, f''(x) = 6(6-1) = 30 \Rightarrow f''$ is positive.

When $x = -2, f''(x) = 6(-4-1) = -30 \Rightarrow f''$ is negative.

Thus $x = 3$ gives minimum and $x = -2$ gives maximum.

Maximum value $f(-2) = 54$.

Minimum value $f(3) = -71$.

2) Show that the least value of $a^2\sec^2 x + b^2\operatorname{cosec}^2 x$ is $(a+b)^2$.

Solution:

Let $f(x) = a^2\sec^2 x + b^2\operatorname{cosec}^2 x$

$$\Rightarrow f'(x) = 2\frac{a^2\sin^4 x - b^2\cos^4 x}{\cos^3 x\sin^3 x}$$

At the maximum or minimum, $f'(x) = 0$.

$$\& \ f''(x) = \frac{8\sin^4 x\cos^4 x\,(a^2\sin^4 x + b^2\cos^4 x)}{\cos^6 x\sin^6 x} = +ve\ expression.$$

$$\Rightarrow a^2\sin^4 x + b^2\cos^4 x = 0 \text{ gives a minimum.}$$

$$\Rightarrow \tan^2 x = \frac{b}{a}.$$

The least value of $f(x)$ is given when $\tan^2 x = \frac{b}{a}$.

$$f(x) = a^2\sin^2 x + b^2\cos^2 x$$

$$= a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right) = (a+b)^2.$$

3) The greatest value of $ax + by$ where x & y are positive and $x^2 + xy + y^2 = 3k^2$ is $2k\sqrt{a^2 - ab + b^2}$.

Solution:

Let $u = ax + by$.

u attains a maximum or a minimum when $\frac{du}{dx} = 0$ & $\frac{d^2y}{dx^2}$ is –ve or +ve.

$$\text{Now } a + b\frac{dy}{dx} = 0 \qquad \text{-----(1)}$$

$$x^2 + xy + y^2 = 3k^2.$$

Differentiating the above equation, $(2x + y) + (x + 2y)\frac{dy}{dx} = 0$ -----(2)

Equating the two values of $\frac{dy}{dx}$, we get $-\frac{a}{b} = -\frac{2x+y}{x+2y}$.

Solving for y, $y = \frac{a-2b}{b-2a}x$

Differentiating equation (2) once again, we get

$$2 + 2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x + 2y)\frac{d^2y}{dx^2} = 0.$$

Substituting the values of $\frac{dy}{dx}$ and y from (1) and (3), we get,

$$\frac{d^2y}{dx^2} = \frac{2}{3}\frac{a^2 - ab + b^2}{b^2}\frac{b - 2a}{x}.$$

$\frac{d^2y}{dx^2}$ is negative for a maximum.

$\frac{b-2a}{x}$ is –ve since $\frac{a^2-ab+b^2}{b^2}$ is +ve.

$x^2 + xy + y^2 = 3k^2.$

Substituting the value for y from (3), we get,

$$x\sqrt{a^2 - ab + b^2} = -k(b - 2a).$$

We take the negative sign, since $\frac{b-2a}{x}$ is –ve.

$$ax + by = ax + \frac{b(a-2b)}{(b-2a)}x$$

$$= -2(a^2 - ab + b^2)\frac{x}{b-2a}$$

$$= 2k\sqrt{a^2 - ab + b^2}.$$

I. Hyperbolic Functions:

If θ is expressed is radians, $\cos\theta$ and $\sin\theta$ can be expanded in powers of θ, the results being

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \infty$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \infty$$

If $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \infty$ -----(1)

Put $x = i\theta$ in the (1). Then

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \infty = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} + \cdots \infty$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \infty\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \infty\right) = \cos\theta + i\sin\theta.$$

$$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta.$$

This is known as ***Euler's formula***.

Similarly, put $x = -i\theta$ in the (1). Then $e^{-i\theta} = \cos\theta - i\sin\theta$.

Note:

(1) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

(2) $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

(3) $\cosh x = \frac{e^x+e^{-x}}{2}$.

(4) $\sinh x = \frac{e^x-e^{-x}}{2}$.

(5) $\tanh x = \frac{\sinh x}{\cosh x}$.

(6) $\operatorname{sech} x = \frac{1}{\cosh x}$.

(7) $\operatorname{cosech} x = \frac{1}{\sinh x}$.

(8) $\coth x = \frac{1}{\tanh x}$.

II. Relations between Hyperbolic functions:

(1) $\cosh^2 x - \sinh^2 x = \frac{1}{4}\{(e^x + e^{-x})^2 - (e^x - e^{-x})^2\} = 1$.

(2) $2\ \sinh x\ \cosh x = 2.\left(\frac{e^x-e^{-x}}{2}\right).\left(\frac{e^x+e^{-x}}{2}\right) = \sinh x$.

(3) $\cosh^2 x + \sinh^2 x = \cosh 2x$.

(4) $\cosh 2x = 2\cosh^2 x - 1$.

(5) $\cosh 2x = 1 + \sinh^2 x$.

(6) $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$.

(7) $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$.

(8) The series for $\sinh x$ and $\cosh x$ are derived below:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \infty$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \infty$$

Subtracting $e^x - e^{-x} = 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \infty\right)$

$$\Rightarrow \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \infty$$

Adding $e^x + e^{-x} == 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \infty\right)$

$$\Rightarrow \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \infty$$

(9) Consider $\sin^2\theta + \cos^2\theta = 1$. Put $\theta = ix$.

$$\sin^2 ix + \cos^2 ix = 1$$

$$\Rightarrow (i\sinh x)^2 + (\cosh x)^2 = 1 \Rightarrow \cosh^2 x - \sinh^2 x = 1.$$

(10) Consider $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Put $\theta = ix$. Then

$$\cos 2ix = \cos^2 ix - \sin^2 ix = (\cosh x)^2 - (i\sinh x)^2 = \cosh^2 x + \sinh^2 x.$$

(11) Consider $\sin 2\theta = 2\sin\theta \,.\cos\theta$

Put $\theta = ix$. Then

$$\sin 2ix = 2\sin ix \,.\cos ix \Rightarrow i\ \sinh 2x = 2i\ \sinh x\cosh x$$

$$\Rightarrow \sinh 2x = 2\sinh x\cosh x.$$

(12) Consider $1 + \tan^2\theta = \sec^2\theta$

Put $\theta = ix$. Then

$$1 + \tan^2 ix = \sec^2 ix \Rightarrow 1 + (i\ \tanh^2 x) = \text{sech}^2\, x$$

III. Inverse Hyperbolic Functions:

We can express $\sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x$ in terms of the logarithmic functions:

(1) $y = \sinh^{-1} x$. Then $x = \sinh y$

$$\frac{e^y - e^{-y}}{2} = x \Rightarrow e^{2y} - 1 = 2xe^y \Rightarrow e^{2y} - 2xe^y - 1 = 0.$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since e^y is positive $e^y = x \pm \sqrt{x^2 + 1}$.

Taking logarithms to the base "e" on both sides, we have,

$$y = \log_e \left(x \pm \sqrt{x^2 + 1}\right)$$

$$\sinh^{-1} x = \log_e \left(x \pm \sqrt{x^2 + 1}\right).$$

(2) $y = \cosh^{-1} x$. Then $x = \cosh y$

$$\frac{e^y + e^{-y}}{2} = x \Rightarrow e^{2y} - 1 = 2xe^y \Rightarrow e^{2y} - 2xe^y - 1 = 0.$$

$$\Rightarrow e^y = x \pm \sqrt{x^2 - 1} = x + \sqrt{x^2 - 1} \; or \; x - \sqrt{x^2 - 1}$$

Thus $y = \log_e\left(x + \sqrt{x^2 - 1}\right)$ or $y = \log_e\left(x - \sqrt{x^2 - 1}\right)$.

The positive sign is usually taken.

Hence $y \cosh^{-1} x. = \log_e\left(x + \sqrt{x^2 - 1}\right)$.

(3) $y = \tanh^{-1} x$. Then $x = \tanh y$.

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x \Rightarrow e^y - e^{-y} = x(e^y + e^{-y}) \Rightarrow e^y(1 - x) = e^{-y}(1 + x)$$

$$\Rightarrow e^{2y} = \frac{1 + x}{1 - x} \Rightarrow 2y = \log_e \left(\frac{1 + x}{1 - x}\right) \Rightarrow y = \frac{1}{2}\log_e \left(\frac{1 + x}{1 - x}\right)$$

$$\Rightarrow \tanh^{-1} x = \frac{1}{2}\log_e \left(\frac{1 + x}{1 - x}\right).$$

Solved Problems:

(1) If $\tan A = \tan\alpha \tan\beta$, $\tan B = \cot\alpha \tanh\beta$, then prove that

$\tan(A + B) = \sinh 2\beta \cos 2\alpha$.

Solution:

Given: $\tan A = \tan\alpha \tan\beta$, $\tan B = \cot\alpha \tanh\beta$

Now $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan\alpha \tan\beta + \cot\alpha \tanh\beta}{1 - \tan\alpha \tan\beta . \cot\alpha \tanh\beta} = \frac{\tanh\beta(\tan\alpha + \cot\alpha)}{1 - \tanh^2\beta}$

$$= \frac{\sinh\beta \cosh\beta}{\cosh^2\beta - \sinh^2 x}\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right) = \frac{\sinh\beta . \cosh\beta}{\sin\alpha . \cos\alpha} = \sinh 2\beta . \operatorname{cosec} 2\alpha.$$

(2) If $\cos(x+iy)=\cos\theta+i\sin\theta$, then prove that $\cos 2x+\cosh 2y=2$.

Solution:

$$\text{Given: } \cos\theta+i\sin\theta=\cos(x+iy)=\cos x\cos(iy)-\sin x\sin(iy)$$
$$=i\cos x\cos y-i\sin x\sin y$$

Equating the real and imaginary parts, we have,

$$\cos\theta=\cos x\cosh y\ \ \&\sin\theta=-\sin x\sinh y.$$

Squaring and adding,

$$\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y=1\Rightarrow\cos^2 x\cosh^2 y+(1-\cos^2 x)\sinh^2 y=1$$
$$\Rightarrow\cos^2 x\,(\cosh^2 y-\sinh^2 x)\sinh^2 y=1\Rightarrow\cos^2 x+\sinh^2 y=1$$
$$\Rightarrow\frac{1+\cos 2x}{2}+\frac{\cosh 2y-1}{2}=1\Rightarrow\cos 2x+\cosh 2y=2.$$

(3) Separate into real and imaginary parts $\tanh(1+i)$.

Solution:

$$\tan ix=i\tanh x$$

Put $x=1+i$.

$$\tanh(1+i)=\tan 1(1+i)=\tan(i-1).$$
$$\Rightarrow i\tanh(1+i)=\frac{\sin(i-1)}{\cos(i-1)}=\frac{2\cos(i+1)\sin(i-1)}{2\cos(i+1)\cos(i-1)}=\frac{\sin(2i)-\sin 2}{\cos(2i)+\cos 2}$$
$$=\frac{i\sinh 2-\sin 2}{\cosh 2+\cos 2}\Rightarrow\tanh(1+i)=\frac{\sinh 2+i\sin 2}{\cosh 2+\cos 2}.$$

(4) If $\tan(x+iy)=u+iv$, then prove that $\frac{u}{v}=\frac{\sin 2x}{\sinh 2y}$.

Solution:

$$\tan(x+iy)=\frac{\sin(x+iy)}{\cos(x+iy)}=\frac{2\cos(x-iy)\sin(x+iy)}{2\cos(x-iy)\cos(x+iy)}=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

This expression is given as $u+iv$

Thus $u=\frac{\sin 2x}{\cos 2x+\cosh 2y}$ & $v=\frac{\sinh 2y}{\cos 2x+\cosh 2y}\Rightarrow\frac{u}{v}=\frac{\sin 2x}{\sinh 2y}$.

Summation of Trignometrical Series

I. Method of difference

When the r^{th} term of a trignometrical series can be expressed as the difference of two quantities, one of which is the same function of r as the other is of $r+1$, the sum of the series may be found as illustrated in the following examples:

Solved Problems:

(1) Find the sum of the series $\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}2^2\theta \ldots + \operatorname{cosec}2^{n-1}\theta$.

Solution:

We have the identity $\operatorname{cosec}\theta = \cot\frac{1}{2}\theta - \cot\theta$.

Similarly $\operatorname{cosec}2\theta = \cot\theta - \cot 2\theta$

$\operatorname{cosec}2^2\theta = \cot\theta - \cot 2^2\theta$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\operatorname{cosec}2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta$

By addition, the required sum $= \cot\frac{1}{2}\theta - \cot 2^{n-1}\theta$

(2) Find the sum of the series $\tan^{-1}\frac{x}{1+1.2x^2} + \tan^{-1}\frac{x}{1+2.3x^2} + \cdots + \tan^{-1}\frac{x}{1+n(n+1)x^2}$.

Solution:

Here $T_r = \tan^{-1}\frac{x}{1+r(r+1)x^2}$

$= \tan^{-1}\frac{(r+1)x-rx}{1+r(r+1)x^2}$

$T_r = \tan^{-1}(r+1)x - \tan^{-1}rx$.

Putting $r = 1,2,3,\ldots,n$, we have

$T_1 = \tan^{-1}2x - \tan^{-1}x$.

$T_2 = \tan^{-1}3x - \tan^{-1}2x$.

$T_3 = \tan^{-1}4x - \tan^{-1}3x$.

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$T_n = \tan^{-1}(n+1)x - \tan^{-1}nx$.

By addition, $S_n = \tan^{-1}(n+1)x - \tan^{-1}x$.

II. Sum of the series of n angles in A. P

Solved Problems:

(1) Find the sum of the series $\cos^2 x+\cos^2(x+y)+\cos^2(x+2y)+\cdots+$up to n terms.

Solution:

Since $\cos^2 x=\frac{1}{2}(1+\cos 2x); \cos^2(x+y)=\frac{1}{2}\left(1+\cos\overline{(2x+2y)}\right)$ ... etc.

$$S_n=\frac{1}{2}(1+\cos 2x)+\frac{1}{2}\left(1+\cos\overline{(2x+2y)}\right)+\cdots+\frac{1}{2}\left(1+\cos\overline{(2x+(2n-1)y)}\right)$$

$$=\frac{n}{2}+\frac{1}{2}(\cos 2x+\cos(2x+2y))+\cdots+\frac{1}{2}\left(1+\cos\overline{(2x+(2n-1)y)}\right)$$

$$=\frac{n}{2}+\frac{1}{2}\frac{\sin ny}{\sin y}\left(1+\cos\overline{(2x+(2n-1)y)}\right).$$

(2) Find the sum of the series $\cosh^2 x+\cosh^2(x+y)+\cosh^2(x+2y)+\cdots+$up to n terms.

Solution:

$\cosh(i\theta)=\cosh\theta.$

$$S_n=\cos ix+\cos i(x+y)+\cos i(x+2y)+\cdots$$

$$=\cos\alpha+\cos i(\alpha+\beta)+\cos i(\alpha+2)+\cdots \text{ to n terms where}$$

$$\alpha=ix \text{ and } \beta=iy$$

$$=\frac{\cos\left(x+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}=\frac{\cos\left(ix+\frac{n-1}{2}iy\right)\sin\frac{niy}{2}}{\sin\frac{iy}{2}}=\frac{\cos\left(ix+\frac{n-1}{2}y\right)\sin\frac{ny}{2}}{\sin\frac{y}{2}}.$$

III. Gregory's Series

To prove that, if θ lies between $\pm\frac{\pi}{4}$, $\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\cdots$

$$e^{i\theta}=\cos\theta+i\sin\theta=\cos\theta\,(1+i\tan\theta).$$

Taking log on both the sides,

$$i\theta=\log\cos\theta+\log(1+i\tan\theta)=\log\cos\theta+i\tan\theta-\frac{i^2\tan^2\theta}{2}+\frac{i^3\tan^3\theta}{2}-\cdots$$

As $|i\tan\theta|=|\tan\theta|<1$ since θ lies between $\pm\frac{\pi}{4}$.

$$i\theta=\log\cos\theta+\log(1+i\tan\theta)=\log\cos\theta+i\tan\theta-\frac{\tan^2\theta}{2}-\frac{i\tan^3\theta}{2}-\cdots$$

Equating the imaginary parts,

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$$

Corollary:1

The above series can be transformed by putting $\tan\theta = x$ so that x is numerically not greater than 1. Then $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ where $\tan^{-1}x$ lies between $\pm\frac{\pi}{4}$.

Corollary:2

More generally, if θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$, when n is an integer, then

$\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$

Put $\theta = n\pi + \varphi$, then φ lies between $\pm\frac{\pi}{4}$.

Arguing as before,

$$\varphi = \tan\varphi - \frac{\tan^3\varphi}{3} + \frac{\tan^5\varphi}{5} - \cdots$$

Substituting $\varphi = n\pi - \theta, \theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots$

Corollary:3 Value of π.

Putting $x = 1$ in $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots = 1 - \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{7} - \frac{1}{9}\right) - \cdots$$

assuming that the series can be arranged.

$$\frac{\pi}{4} = 1 - \left\{2\left(\frac{1}{3.5}\right) + \left(\frac{1}{7.9}\right) + \cdots\right\}$$

This series can be used to calculate π. But the defeat with this that successiveterms do not decrease rapidly. Hence a large number of terms in the above expansion have to be taken to obtain a fairly accurate value of π.